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The influence of the inertial properties of the parts of gimbals on the dynamics of a rigid body ☆

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ABSTRACT

The problems of the motion of two rigid bodies, namely, a uniform thin rod with a fixed end in a gravitational field and a uniform flat circular disk with a fixed centre, are considered. The behaviour of these bodies as a function of the method used to implement the constraint, namely, using an ideal ball joint or different versions of gimbals, are compared. The features of the motion specified by the inertial properties of the parts of the gimbals are found ©2009.

Suppose one of the ends of a thin uniform rod (a section of a straight line) with length $2l$ and weight mg is fixed using an ideal ball joint. We will choose as the generalized coordinates of this rod the angles ϑ $(0 < \vartheta < \pi/2)$ and φ $(0 \le \varphi < 2\pi)$, respectively, between the rod and the direction of the force of gravity and between two vertical planes that pass through the fixed point, one of which is fixed and the other contains the rod.

The structure of the Lagrangian in these coordinates

$$L = \frac{2}{3}ml^2(\dot{\vartheta}^2 + \dot{\varphi}^2\sin^2\vartheta) + mgl\cos\vartheta$$

indicates the presence of two first integrals: the energy integral

$$\frac{2}{3}ml^2(\dot{\vartheta}^2 + \dot{\varphi}^2\sin^2\vartheta) - mgl\cos\vartheta = h$$

and the cyclic integral

$$K_z \equiv \frac{4}{3}ml^2\dot{\varphi}\sin^2\vartheta = p$$

(K_z is the projection of the angular momentum of the system onto the Oz axis, which is directed along the force of gravity, see Fig. 1).

These integrals specify two types of motion of the rod:

a) if $\dot{\varphi}(0) = 0$, the rod moves in a fixed vertical plane as an ordinary physical pendulum
b) if $\dot{\varphi}(0) = 0$, then $\dot{\varphi}(t) > 0$ at any time, and $\vartheta(t)$ is a periodic function of time.

Now consider the behaviour of the rod in the case where a Cardan–Hooke universal joint is used to fix its end (Fig. 1).

Suppose the plane of the fixed yoke is vertical and the axis of rotation of the cross piece is horizontal. For simplicity, we will assume that the cross piece is a set of two identical uniform thin rods (sections of a straight line), which are rigidly joined at their midpoints at a right angle, and that the depth a of the movable yoke of the cross piece is equal to 0, so that the end of the rod fastened to this yoke is fixed.

In this case, the constraints applied to the rod will be exactly the same as in the case considered previously, and if the mass of the centre cross is equal to zero, there will be no special features in its motion under the action of gravity.

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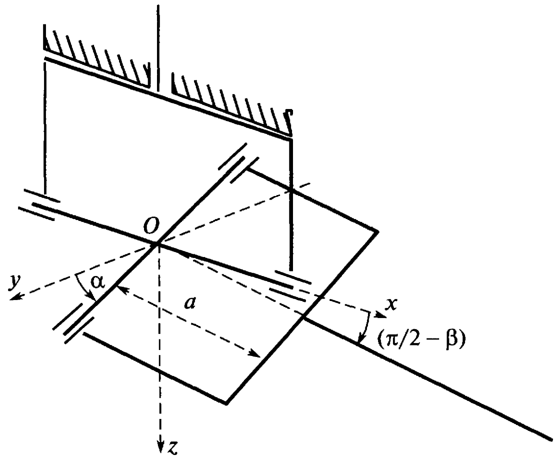

Fig. 1.

Let J $\left(J=\frac{4}{3}ml^2\right)$ be the moment of inertia of the rod about the axis that passes through its end and is perpendicular to it, and let the moment of inertia of the cross piece about its axis of rotation be equal to εJ ($\varepsilon \ll 1$).

It is convenient to choose the angles α (the angle of rotation of the cross piece, $-\pi/2<\alpha<\pi/2$) and β (the angle between the rod and the plane that passes through the fixed point and is perpendicular to the axis of rotation of the cross piece, $-\pi/2<\beta<\pi/2$) as the coordinates in this case (Fig. 1).

We write the Lagrangian of the system consisting of the rod and the cross piece:

$$L_1=\frac{1}{2}J(\dot{\alpha}^2\cos^2\beta+\dot{\beta}^2)+\frac{1}{2}\varepsilon J\dot{\alpha}^2+mgl\cos\alpha\cos\beta$$

This very simple working model of the system exhibits its remarkable behaviour. If we set

$$\alpha(0)=\alpha_0>0,\beta(0)=\beta_0>,\dot{\alpha}(0)=0,\dot{\beta}(0)=0$$

the rod at first moves near the vertical plane passing through the initial point. Then the angle of deflection from the vertical plane gradually increases, and the rod begins to rotate about the vertical. The rotation gradually changes to planar oscillations, whose plane differs from the original plane. After a certain number of oscillations, the rod again begins to rotate about the vertical, but now in the opposite direction, and so forth. To explain the phenomenon observed, we will use the theory of small oscillations.

In the specified region where the angles α and β are defined, the system of equilibrium equations

$$\frac{\partial U}{\partial\alpha}=-mgl\sin\alpha\cos\beta=0,\quad \frac{\partial U}{\partial\beta}=-mgl\cos\alpha\sin\beta=0$$

has a single solution: $\alpha=0$, $\beta=0$.

The matrices of the kinetic energy A and the second derivatives of the potential energy B in this equilibrium position have the form

$$A=\operatorname{diag}\|\,J(1+\varepsilon),J\,\|,B=\operatorname{diag}\|\,mgl,mgl\,\|$$

After writing the characteristic equation

$$\det(-\lambda^2A+B)=0,$$

we find its roots

$$\lambda_1^2=\frac{mgl}{J(1+\varepsilon)},\quad \lambda_2^2=\frac{mgl}{J}\Rightarrow\lambda_1\approx\left(1-\frac{\varepsilon}{2}\right)\lambda_2$$

The forms of the matrices A and B suggest that α and β are normal coordinates; therefore, the solution of the problem of small oscillations under the initial conditions indicated above has the form

$$\alpha(t)=\alpha_0\cos\lambda_1 t,\quad \beta(t)=\beta_0\cos\lambda_2 t.$$

The projection of the angular momentum of the system onto the vertical axis is the sum of the projections of the angular momentas of the cross piece and the rod onto this axis. As a consequence of the constraints imposed on the cross piece, the vertical component of its angular momentum is equal to zero. This means that the projection of the angular momentum of the system onto the vertical is identical to the corresponding projection of the angular momentum of the rod. Therefore, if this projection is non-zero and its sign remains unchanged over a time interval exceeding the longest of the periods of small oscillations found, the rod will rotate about the vertical in this time interval, i.e., it will go around it on all sides without changing its direction of rotation. Let ρ be the distance between a certain point on the rod and the centre of the cross piece, and let $x = \rho \sin \beta$ and $y = (-\rho \cos \beta \sin \alpha)$ be the coordinates of the projection of this point onto the xOy plane. Then for the projection of the angular momentum of the rod onto the Oz plane in the variables (α, β), we have

$$K_z = \int_0^{2l} \frac{m}{2l}(x\dot{y} - \dot{x}y)d\varrho = J(-\dot{\alpha}\cos\alpha\cos\beta\sin\beta + \dot{\beta}\sin\alpha)$$

Considering the expression obtained within the theory of small oscillations (under assigned initial conditions), we find

$$K_z \approx J(-\dot{\alpha}\beta + \alpha\dot{\beta}) = J\alpha_0\beta_0(\lambda_1 \sin\lambda_1 t\cos\lambda_2 t - \lambda_2\cos\lambda_1 t\sin\lambda_2 t)$$

If the mass of the cross piece is equal to zero, we have

$$\varepsilon = 0 \Rightarrow \lambda_1 = \lambda_2 \Rightarrow K_z = 0$$

as we should when all the circumstances mentioned are taken into account. If $\varepsilon \neq 0$ and the frequencies of the small oscillations are different, simple reduction converts the latter formula into the form

$$\begin{aligned} K_z &\approx J\alpha_0\beta_0((\lambda_+ + \lambda_-)\sin\lambda_1 t\cos\lambda_2 t - (\lambda_+ - \lambda_-))\cos\lambda_1 t\sin\lambda_2 t = \\ &= J\alpha_0\beta_0(\lambda_+\sin 2\lambda_- t + \lambda_-\sin 2\lambda_+ t) = \\ &= J\alpha_0\beta_0\left(-\lambda_2\left(1-\frac{\varepsilon}{4}\right)\sin\frac{\lambda_2\varepsilon}{2}t - \frac{\lambda_2\varepsilon}{4}\sin 2\lambda_2\left(1-\frac{\varepsilon}{2}\right)t\right); \quad \lambda_\pm = \frac{\lambda_1 \pm \lambda_2}{2} \end{aligned} \tag{1}$$

Formula (1) describes the main features of the observed motion of the rod. At the beginning of its motion, the projection of the angular momentum onto the vertical is close to zero, and the rod oscillates, almost without deviating from the fixed vertical plane. At a certain time the first term becomes greater in absolute value than the second term, and the sign of K_z remains unchanged over a fairly long time interval. During this time interval the rod rotates about the vertical in a definite direction (see the expression for K_z in the variables (ϑ, φ)).

However, the value of K_z subsequently becomes close to zero, and the rod oscillates near a fixed vertical plane, which differs from the original vertical plane. Next, the first term begins to become dominate again, and the rod begins to rotate about the vertical, but in the opposite direction, since the sign of K_z has changed. The pattern then repeats itself.

The region of applicability of formula (1) can be extended by introducing a correction to the frequency. Observations of the behaviour of the model reveal that the qualitative features of the motion are maintained for any initial conditions, excluding the initial conditions

$$(\alpha(0) \neq 0, \dot{\alpha}(0) \neq 0, \beta(0) = 0, \dot{\beta}(0) = 0,), \quad (\alpha(0) = 0, \dot{\alpha}(0) = 0, \beta(0) \neq 0, \dot{\beta}(0) \neq 0)$$

under which the rod oscillates in the corresponding fixed vertical planes.

The problem considered is an example of how a change in the method used to implement the constraint (the set of virtual displacements of the rod for both suspension methods is the same), which is associated with a "small" change in the form of kinetic energy of the system, leads to fundamental changes in the set of actual motions.

Another example of how a change in the method used to implement the constraint can influence the behaviour of the system is provided by the problem of the motion of a uniform circular disk with a fixed centre. We will first assume that the centre of the disk is fixed using an ideal ball joint. Then the problem reduces to Euler's case of the motion of a dynamically symmetrical rigid body with a fixed point. As we know, the motion in this case is a regular precession, under which the axis of dynamic symmetry of the body (henceforth the axis of the disk) describes a right circular cone about a fixed angular momentum vector. If the direction of the angular velocity vector is initially identical to the direction of its axis, the angular momentum vector will be directed along this axis; therefore, the axis of the disk will be fixed in inertial space. Hence, if this device could be implemented, it could serve as a navigational instrument.

If there is misalignment under the assigned initial conditions, i.e., if the direction of the angular velocity vector is not the same as the direction of the axis of the disk, but the angle between the axis of the disk and the angular momentum vector is small, this angle will remain small during the entire time of motion as a consequence of the properties of regular precession. Hence, in this case, too, knowing the direction of the axis of the disk, we can determine the fixed direction in inertial space with obvious reservations.

Now suppose the ideal constraints applied to the disk are realized using gimbals (see Fig. 2).

We will assume that a weightless rod passes through the centre of the disk perpendicular to its plane and specifies the axis of rotation of the disk $O\zeta$ relative to the inner gimbal. Suppose the axis of the outer gimbal of the suspension is vertical, the axis of the inner gimbal is horizontal, and the centres of gravity of the disk and each of the gimbals coincide with the centre of the suspension.

We will use α, β and γ to denote the angles of rotation of the outer gimbal relative to the fixed trihedron, the inner gimbal relative to the outer gimbal, and the disk relative to the inner gimbal, respectively. We will use the letter O to denote the centre of the suspension, and we will bind the movable trihedron $Oxyz$ to the outer gimbal (Fig. 2). We introduce the movable trihedron $O\xi\eta\zeta$, which is bound to the inner gimbal in such a manner that the following relations hold for its unit vectors: $\mathbf{e}_x = \mathbf{e}_\xi$, $(\mathbf{e}_y, \mathbf{e}_\zeta) = \cos\beta$. We will assume that the outer and inner gimbals are identical and that the following relations hold for the moments of inertia (about the principal axes) of the outer gimbal (J_x, J_y, J_z), the inner gimbal (J_ξ, J_η, J_ζ), and the disk (A, B, C)

$$J_z = J_x = J, \quad J_y = 2J, \quad J_\zeta = J_\xi = J, \quad J_\eta = 2J, \quad A = B = C/2$$

We write the expressions for the angular velocities ω_1, ω_2 and ω_3 of the outer gimbal, the inner gimbal and the disk, respectively,

$$\boldsymbol{\omega}_1 = \dot{\alpha}\mathbf{e}_z, \quad \boldsymbol{\omega}_2 = \dot{\alpha}\mathbf{e}_z + \dot{\beta}\mathbf{e}_\xi, \quad \boldsymbol{\omega}_3 = \dot{\alpha}\mathbf{e}_z + \dot{\beta}\mathbf{e}_\xi + \dot{\gamma}\mathbf{e}_\zeta$$

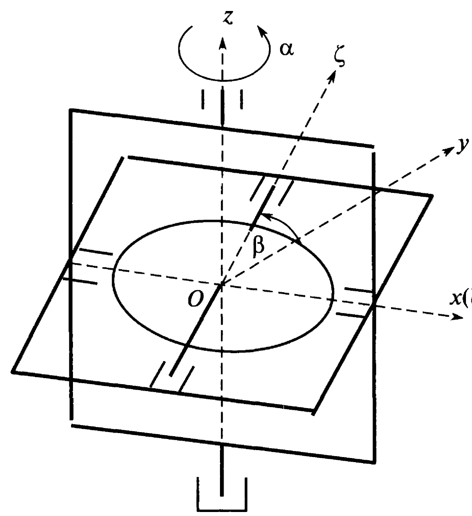

Fig. 2.

Accordingly, the kinetic energy of each of these bodies has the form

$$T_1 = \frac{1}{2}J\dot{\alpha}^2, \quad T_2 = \frac{1}{2}J(\dot{\beta}^2 + \dot{\alpha}^2(1+\cos^2\beta)), \quad T_3 = \frac{1}{4}C\left(\dot{\beta}^2 + \dot{\alpha}^2\cos^2\beta + 2(\dot{\alpha}\sin\beta + \dot{\gamma})^2\right)$$

and the kinetic energy of the system is

$$T = T_1 + T_2 + T_3 = \frac{1}{2}C\left(\Delta(\beta)\dot{\alpha}^2 + \lambda\dot{\beta}^2 + (\dot{\alpha}\sin\beta + \dot{\gamma})^2\right)$$

$$\Delta(\beta) = \mu + \lambda\cos^2\beta, \quad \mu = \frac{2J}{C}, \quad \lambda = \frac{2J+C}{2C}$$

The structure of the Lagrangian, which is the same as the kinetic energy in the problem under consideration, enables us to write the three first integrals (the energy integral and two cyclic integrals):

$$T = h \Rightarrow \frac{1}{2}C\left(\Delta(\beta)\dot{\alpha}^2 + \lambda\dot{\beta}^2 + (\dot{\alpha}\sin\beta + \dot{\gamma})^2\right) = h$$
$$\frac{\partial T}{\partial\dot{\alpha}} = p_1 \Rightarrow C\left(\Delta(\beta)\dot{\alpha} + (\dot{\alpha}\sin\beta + \dot{\gamma})\sin\beta\right) = p_1, \quad \frac{\partial T}{\partial\dot{\gamma}} = p_2 \Rightarrow C(\dot{\alpha}\sin\beta + \dot{\gamma}) = p_2 \tag{2}$$

The equations of motion of the system specified by these first integrals have the solution

$$\dot{\alpha}(t) \equiv 0, \dot{\beta}(t) \equiv 0, \dot{\gamma}(t) \equiv \omega_0, \alpha(t) = \alpha_0, \beta(t) = \beta_0, \gamma(t) = \gamma_0 + \omega_0 t$$

for any α_0, β_0, γ_0 and ω_0, i.e., during translational motion of the object on which the suspension is mounted, the axis of the disk maintains its direction in fixed space.

The question now arises of how the system will behave if we slightly disturb the initial conditions

$$\dot{\alpha}(0) = \alpha_0, \dot{\beta}(0) = \dot{\beta}_0, \dot{\gamma}(0) = \omega_0 \gg \dot{\beta}_0, \alpha(0) = \alpha_0, \beta(0) = \beta_0, \gamma(0) = \gamma_0$$

In order to answer it, we will rewrite the first integrals found, taking into account the new initial conditions after replacing the right-hand sides of equalities (2), respectively, by

$$\frac{1}{2}C(\lambda\dot{\beta}_0^2 + \omega_0^2), \quad C\omega_0\sin\beta_0, \quad C\omega_0$$

Using the last equation to eliminate the variable $\dot{\gamma}$ from the first two equations, we find

$$\frac{1}{2}\left(\Delta(\beta)\dot{\alpha}^2 + \lambda\dot{\beta}^2\right) = \frac{1}{2}\lambda\dot{\beta}_0^2, \quad \dot{\alpha} = \omega_0\frac{\sin\beta_0 - \sin\beta}{\Delta(\beta)} \tag{3}$$

Substituting the expression for $\dot{\alpha}$ from the second equality in (3) into the first, we obtain the equation for determining β

$$\frac{(\sin\beta_0 - \sin\beta)^2}{2\Delta(\beta)} + \frac{\lambda}{2}\left(\frac{\dot{\beta}}{\omega_0}\right)^2 = \frac{\lambda}{2}\varepsilon^2; \quad \varepsilon = \frac{\dot{\beta}_0}{\omega_0} \ll 1 \tag{4}$$

It can be seen from Eq. (4) that $\beta(t)$ is a periodic function and that

$$|\beta(t)-\beta_0| \ll 1$$

We make the replacement of variables

$$\beta = \beta_0 + q$$

The equation for determining q has the form

$$\ddot{q} + \frac{\omega_0^2}{\lambda} V'(q) = 0; \quad V(q) = \frac{(\sin\beta_0 - \sin(\beta_0+q))^2}{2\Delta(\beta_0+q)} \tag{5}$$

To find the solution of Eq. (5), we use the method of Lindstedt and Newcomb, whose procedure was described in detail in Ref. 1. Suppose following the conditions holds at the time t_0

$$q(t_0) = a > 0, \quad \dot{q}(t_0) = 0$$

Solving the equation

$$\frac{(\sin\beta_0 - \sin\beta)^2}{\Delta(\beta)} = \lambda\varepsilon^2$$

we find

$$\sin\beta = \frac{\sin\beta_0}{1+(\lambda\varepsilon)^2} \pm \frac{\lambda\varepsilon}{1+(\lambda\varepsilon)^2}\sqrt{1+\frac{\mu}{\lambda}-\sin^2\beta_0+(\lambda\varepsilon)^2\left(1+\frac{\mu}{\lambda}\right)}$$

whence we obtain the expression for the amplitude

$$a = \frac{\lambda\varepsilon}{\cos\beta_0}\sqrt{1+\frac{\mu}{\lambda}-\sin^2\beta_0} + O((\lambda\varepsilon)^2)$$

Since the value of $q(t)$ remains small during the entire time of motion, the second term in Eq. (5) can be represented in the form of a convergent series

$$\frac{\omega_0^2}{\lambda} V'(q) = k^2 q(1 + c_1 q + c_2 q^2 + \ldots)$$

After introducing the dimensionless time

$$\vartheta = k(t-t_0)\sqrt{1+\delta}$$

(where δ is a constant, to be determined) Eq. (5) takes the form

$$(1+\delta)\frac{d^2q}{d\vartheta^2} + q = -c_1 q^2 - c_2 q^3 - \ldots \tag{6}$$

We will seek a solution of Eq. (6) in the form of a power series in the amplitude a, which converges for fairly small values of this parameter

$$q(\vartheta) = q_1(\vartheta)a + q_2(\vartheta)a^2 + q_3(\vartheta)a^3 + \ldots \tag{7}$$

with the initial conditions

$$q_1(0) = 1, q_2(0) = 0, q_3(0) = 0, \ldots$$
$$\dot{q}_1(0) = 0, \ \dot{q}_2(0) = 0, \ \dot{q}_3(0) = 0, \ldots$$

Since the oscillation period of a non-linear system depends on the amplitude, we will represent δ in the form

$$\delta = \delta_1 a + \delta_2 a^2 + \delta_3 a^3 + \ldots \tag{8}$$

After substituting expressions (7) and (8) into Eq. (6), expanding the right-hand and left-hand sides of this equation in powers of a and equating corresponding coefficients, we obtain the following system of equations for finding the functions $q_1(\vartheta)$ and the constants δ_i ($i = 1, 2, \ldots$)

$$\frac{d^2q_1}{d\vartheta^2} + q_1 = 0, \quad \frac{d^2q_2}{d\vartheta^2} + q_2 = -\delta_1\frac{d^2q_1}{d\vartheta^2} - c_1 q_1^2, \ldots$$

Under the specified initial conditions the solution of the first equation is

$$q_1 = \cos\vartheta$$

The solution of the second equation for the $q_1(\vartheta)$ obtained will be periodic only if $\delta_1 = 0$. After adopting this condition, we obtain

$$q_2 = c_1\left(-\frac{1}{2} + \frac{1}{3}\cos\vartheta + \frac{1}{6}\cos 2\vartheta\right)$$

Thus, we have (see Ref. 1)

$$q(\vartheta) = -\frac{1}{2}c_1a^2 + O(a^3) + (a + \frac{1}{3}c_1a^2 + O_1(a^3))\cos\vartheta + \\ + \left(\frac{1}{6}c_1a^2 + O_2(a^3)\right)\cos 2\vartheta + O_3(a^3)\cos 3\vartheta + \sum_{k=4}^{\infty} O(a^k)\cos k\vartheta \tag{9}$$

where $O(a^k)$ denotes estimates of the respective coefficients. Continuing this procedure, we can obtain a solution with any degree of accuracy.

We now consider the second formula in (3) and introduce the notation

$$\omega_0 \frac{\sin\beta_0 - \sin(\beta_0 + q)}{\Delta(\beta_0 + q)} = A(q)$$

Taking into account the periodicity and smallness of $q(\vartheta)$, we represent the function $A(q)$ in the form of a convergent series; therefore,

$$\dot{\alpha}(\vartheta) = A(0) + A'(0)q(\vartheta) + \frac{1}{2}A''(0)q^2(\vartheta) + O(q^3(\vartheta))$$

Using expression (9), we find that

$$\dot{\alpha}(\vartheta) = A(0) - A'(0)\frac{c_1}{2}a^2 + \frac{1}{4}A''(0)a^2 + O(a^3) + \sum_{k=1}^{\infty} O(a^k)\cos k\vartheta$$

Since in the problem under consideration we have

$$A(0) = 0, \quad A'(0) = -\omega_0\frac{\cos\beta_0}{\Delta(\beta_0)}, \quad A''(0) = \omega_0\left(\frac{\sin\beta_0}{\Delta(\beta_0)} - \frac{4\lambda\sin\beta_0\cos^2\beta_0}{\Delta^2(\beta_0)}\right)$$

$$c_1 = \frac{V'''(0)}{2V''(0)} = -\frac{3}{2}\operatorname{tg}\beta_0 + \frac{3\lambda\sin\beta_0\cos\beta_0}{\mu + \lambda\cos^2\beta_0}$$

the period-averaged value of the rate of variation of the angle α has the form

$$\langle\dot{\alpha}(\vartheta)\rangle = -\frac{\mu a^2\omega_0\sin\beta_0}{2(\mu + \lambda\cos^2\beta_0)} + O(a^3) \tag{10}$$

This indicates a systematic drift of the axis of the disk away from its initial position, i.e., the motions of the disk under the two methods used to implement an ideal constraint that leaves its centre fixed are fundamentally different.

Formula (10) is a modified version of the formula obtained by Magnus when solving the problem of the drift of a balanced gyroscope in gimbals.[2] This problem and Magnus' formula have been actively discussed in the Russian literature for many years. A detailed bibliography on this problem was presented in D. M. Klimov's supplement to Ye. L. Nikolai's book.[3] Various proofs of Magnus' formula, which differ from the one presented above, and additional bibliographical references can be found in Refs 4 and 5.

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